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LETTER TO THE EDITOR

q-deformed oscillator algebra as a quantum group

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Abstract. It is proved that the q-deformed oscillator algebra is a quantum group. The Hopf algebraic structure is set up and verified to be consistent and compatible. The universal $\mathcal{R}$-matrix and Yang-Baxter equation are constructed. The q-differential operator algebra is briefly discussed.

A quantum group is a Hopf algebra which is neither commutative nor co-commutative [1-5]. Most of the well studied quantum groups are the q-deformations from semisimple Lie algebras, which revert to semisimple Lie algebras when $q \to 1$. They are deeply rooted in many physics theories, such as exactly solvable statistical models [6], integrable field theories, two-dimensional quantum field theories involving fractal statistics, and conformal field theories [7].

Recently, many works have been devoted to the q-deformed oscillator realization of the quantum algebras [9-26]. This realization supplies the q-deformation of the semisimple Lie algebras such as A_N and C_N and super Lie algebras, and is helpful in investigating their representations. Hence the q-deformed oscillator algebra is a powerful tool in the studies of the quantum algebras.

Let us recall that the physical system [14-19] of a single q-deformed oscillator is described by three operators, the creation and annihilation operators $a_q^\dagger$ and a_q and N. When $q \to 1$, $a_q^\dagger$, $a_q \to a^\dagger$, a, the creation and annihilation operators for the simple harmonic oscillator (SHO). For generic q, define $N \triangleq \lim_{q\to 1} a_q^\dagger a_q + \frac{1}{2} = a^\dagger a + \frac{1}{2}$. The algebraic relations satisfied by these operators are [13, 23]

$$\begin{aligned} &[a_q, a_q^\dagger] = [N+\tfrac{1}{2}]_q - [N-\tfrac{1}{2}]_q \\ &[N, a_q] = -a_q \qquad [N, a_q^\dagger] = a_q^\dagger \end{aligned} \tag{1}$$

where

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}} = \frac{\sinh(\gamma x)}{\sinh \gamma} \qquad \gamma = \ln q. \tag{2}$$

This is $\mathcal{H}_q(1)$, the one-dimensional q-deformed oscillator algebra. When $q \to 1$, $[x] \to x$, the q-deformed algebra reverts to the one-dimensional SHO algebra, denoted $\mathcal{H}(1)$.

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In this letter we show that the q-deformed oscillator algebra is in fact a quantum algebra itself with the supplied Hopf structure. Therefore we find the q-deformation, i.e. the quantum counterpart for the non-semisimple Lie algebra $\mathscr{H}(1)$, which is one of the most important non-semisimple Lie algebras in physics. When $q \to 1$, the quantum algebra $\mathscr{H}_q(1)$ reverts to the non-semisimple Lie algebra $\mathscr{H}(1)$.

Furthermore, we find that the universal $\mathscr{R}$-matrix can be constructed from the q-deformed algebra, and then the Yang-Baxter equation is given.

This letter is organized in the following way. First we provide the Hopf algebraic structure, the co-product, antipode and co-unit for the new algebra. We show via direct calculations that they are consistent and compatible with the algebraic relations. Then we now give the $\mathscr{R}$-matrix explicitly and construct the Yang-Baxter equation (without spectral parameter) in a straightforward way. Finally there is a brief discussion on the q-deformed differential operator algebra.

We now give the Hopf structure for the q-deformed oscillator algebra explicitly and verify that it is self-consistent and compatible with the commutation relations.

First, it is pointed out [1-5] that for a given associative algebra A with unit, we call A a Hopf algebra if we can define three operations in A: the co-product Δ, antipode S and co-unit ε:

$$\begin{aligned} &\Delta: A \to A \otimes A && \Delta(ab) = \Delta(a)\Delta(b) \\ &S: A \to A && S(ab) = S(b)S(a) \\ &\varepsilon: A \to \mathscr{C} && \varepsilon(ab) = \varepsilon(a)\varepsilon(b) \end{aligned} \tag{3}$$

where a, b are elements of A, and $\mathscr{C}$ is the field of complex numbers. The operations should be consistent, i.e.

$$\begin{aligned} &(\mathrm{id} \otimes \Delta)\Delta(a) = (\Delta \otimes \mathrm{id})\Delta(a) \\ &m(\mathrm{id} \otimes S)\Delta(a) = m(S \otimes \mathrm{id})\Delta(a) = \varepsilon(a) \cdot 1 \\ &(\varepsilon \otimes \mathrm{id})\Delta(a) = (\mathrm{id} \otimes \varepsilon)\Delta(a) = a \end{aligned} \tag{4}$$

and compatible with the algebraic relations.

So to identify the q-deformed algebra as a Hopf algebra is to find consistent and compatible (non-trivial) definitions for these three operations. They are given in the following:

$$\begin{aligned} &\Delta(N) = N \otimes 1 + 1 \otimes N - (\alpha/\bar{\gamma}) 1 \otimes 1 \\ &\Delta(a_q^\dagger) = (a_q^\dagger \otimes q^{N/2} + \mathrm{i} q^{-N/2} \otimes a_q^\dagger)\, \mathrm{e}^{-\mathrm{i}\alpha/2} \\ &\Delta(a_q) = (a_q \otimes q^{N/2} + \mathrm{i} q^{-N/2} \otimes a_q)\, \mathrm{e}^{-\mathrm{i}\alpha/2} \\ &S(N) = -N + \mathrm{i}(2\alpha/\gamma) \cdot 1 \qquad S(a_q^\dagger) = -q^{1/2} a_q^\dagger \qquad S(a_q) = -q^{-1/2} a_q \\ &\varepsilon(N) = \alpha/\bar{\gamma} \qquad \varepsilon(a_q^\dagger) = \varepsilon(a_q) = 0 \qquad \varepsilon(1) = 1 \end{aligned} \tag{5}$$

where $\alpha = 2k\pi + \pi/2$, $k \in \mathscr{Z}$.

It is easy to check that this set of definitions satisfies the consistency condition (4). Here the calculation is given to show in the following that the first equation holds for $a_q^\dagger$, i.e.

$$(\mathrm{id} \otimes \Delta)\Delta(a_q^\dagger) = (\Delta \otimes \mathrm{id})\Delta(a_q^\dagger). \tag{6}$$

Proof.

$$\begin{aligned}\text{left} &= (\text{id}\otimes\Delta)(a_q^\dagger\otimes q^{N/2}+\mathrm{i}q^{-N/2}\otimes a_q^\dagger)\,\mathrm{e}^{-\mathrm{i}\alpha/2}\\ &=[a_q^\dagger\otimes q^{\Delta(N)/2}+\mathrm{i}q^{-N/2}\otimes\Delta(a_q^\dagger)]\,\mathrm{e}^{-\mathrm{i}\alpha/2}\\ &=(a_q^\dagger\otimes q^{N/2}\otimes^{N/2}+\mathrm{i}q^{-N/2}\otimes a_q^\dagger\otimes q^{N/2}-q^{-N/2}\otimes q^{-N/2}\otimes a_q^\dagger)\,\mathrm{e}^{-\mathrm{i}\alpha}\end{aligned}\tag{7}$$

$$\begin{aligned}\text{right} &= (\Delta\otimes\text{id})(a_q^\dagger\otimes q^{N/2}+\mathrm{i}q^{-N/2}\otimes a_q^\dagger)\,\mathrm{e}^{-\mathrm{i}\alpha/2}\\ &=[(a_q^\dagger\otimes q^{N/2}\otimes q^{N/2}+\mathrm{i}q^{-N/2}\otimes a_q^\dagger\otimes q^{N/2})\,\mathrm{e}^{-\mathrm{i}\alpha/2}\\ &\quad+\mathrm{i}q^{1/2N\otimes 1-1\otimes N/2+(\alpha/\bar{\gamma})1\otimes 1}\otimes a_q^\dagger]\,\mathrm{e}^{-\mathrm{i}\alpha/2}\\ &=\text{left}.\end{aligned}\tag{8}$$

It is also easy to verify that for any two elements a, b we have

$$\Delta([a,b])=[\Delta(a),\Delta(b)].\tag{9}$$

We provide the proof for a_q and $a_q^\dagger$.

Proof.

$$\text{left}=\Delta([a_q,a_q^\dagger])=-\mathrm{i}\,\frac{q^N\otimes q^N-q^{-N}\otimes q^{-N}}{q+q^{-1}}\tag{10}$$

$$\begin{aligned}\text{right}&=[\Delta(a_q),\Delta(a_q^\dagger)],\\ &=\mathrm{i}([a_q\otimes q^{N/2},a_q^\dagger\otimes q^{N/2}]+\mathrm{i}[a_q\otimes q^{N/2},q^{-N/2}\otimes a_q^\dagger]\\ &\quad+[q^{-N/2}\otimes a_q,a_q^\dagger\otimes q^{N/2}]-[q^{-N/2}\otimes a_q,q^{-N/2}\otimes a_q^\dagger]).\end{aligned}\tag{11}$$

Noticing the relations

$$\begin{aligned}a_q^\dagger\,\mathrm{e}^{\pm\gamma N/2}&=q^{\pm 1/2}\,\mathrm{e}^{\pm\gamma N/2}a_q^\dagger\\ a_q\,\mathrm{e}^{\pm\gamma N/2}&=q^{\mp 1/2}\,\mathrm{e}^{\pm\gamma N/2}a_q\end{aligned}\tag{12}$$

then the second and third commutators in (11) are zero. The first and fourth commutators add up to make the right of (10).

Therefore, the q-deformed oscillator algebra with the above Hopf structure is really a Hopf algebra. Since for generic q, the structure is neither commutative nor co-commutative, the algebra is a non-trivial quantum algebra.

If $q\to 1$ the q-deformed oscillator algebra reduces to a non-semisimple Lie algebra, i.e. the SHO algebra.

The existence of the Yang–Baxter equation is a basic characteristic of quantum groups. We have just shown that the q-deformed oscillator algebra is a quantum group, so a natural question arises as how to construct the Yang–Baxter equation from this algebra. Let us start by setting up the $\mathcal{R}$-matrix.

Actually, the algebraic relations (1) are invariant under $q\to q^{-1}$ (or $\gamma\to-\gamma$), so another co-product can be

$$\begin{aligned}\bar{\Delta}(N)&=N\otimes 1+1\otimes N+(\alpha/\bar{\gamma})1\otimes 1\\ \bar{\Delta}(a_q^\dagger)&=(a_q^\dagger\otimes q^{-N/2}+\mathrm{i}q^{N/2}\otimes a_q^\dagger)\,\mathrm{e}^{-\mathrm{i}\alpha/2}\\ \bar{\Delta}(a_q)&=(a_q\otimes q^{-N/2}+\mathrm{i}q^{N/2}\otimes a_q)\,\mathrm{e}^{-\mathrm{i}\alpha/2}\end{aligned}\tag{13}$$

and there is an operator $\mathcal{R}$ acting in algebra $\mathcal{H}_q(1)^{\otimes 2}$ giving the transformation between Δ and $\bar{\Delta}$

$$\mathcal{R}\Delta\mathcal{R}^{-1}=\bar{\Delta} \tag{14}$$

which can be written explicitly in the form

$$\mathcal{R}=q^{1/2N\otimes N-(\alpha/\bar{\gamma})\Delta(N)}\sum_{n\geq 0}\mathrm{i}^n\frac{(1+q^{-1})^n}{[n]_{q^{1/2}}!}q^{-n(n+1)/4}(a_q^{\dagger})^n\otimes q^{-nN/2}a_q^n \tag{15}$$

where the convention

$$[n]_{q^{1/2}}!=[1]_{q^{1/2}}[2]_{q^{1/2}}\ldots[n]_{q^{1/2}} \tag{16}$$

is applied. It is a straightforward calculation to prove (14), using the following relations:

$$[a_q^n, a_q^{\dagger}]=[2N+n-1]_{(+,q^{1/2})}[n]_{q^{1/2}}a_q^{n-1} \tag{17}$$

and

$$[x]_{(+,q)}=\frac{q^x+q^{-x}}{q+q^{-1}}=\frac{\cosh(\gamma x)}{\cosh\gamma}\qquad \gamma=\ln q. \tag{18}$$

$\mathcal{R}$ is the universal $\mathcal{R}$-matrix with the following properties, which can all be verified by direct calculations:

$$\begin{aligned}(\Delta\otimes\mathrm{id})\mathcal{R}&=\mathcal{R}_{13}\mathcal{R}_{23}\\(\mathrm{id}\otimes\Delta)\mathcal{R}&=\mathcal{R}_{13}\mathcal{R}_{12}\\(S\otimes\mathrm{id})\mathcal{R}&=\mathcal{R}^{-1}\end{aligned} \tag{19}$$

where the $\mathcal{R}_{ij}$ are the embedding of $\mathcal{R}$ into $\mathcal{H}_q(1)^{\otimes 3}$. Hence we can show the Yang–Baxter equation stands:

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23}=\mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}. \tag{20}$$

In other words, the Yang–Baxter equation has a solution constructed from quantum enveloping algebra of a non-semisimple Lie algebra $\mathcal{H}_q(1)$. This is an interesting result.

We have shown that the one-dimensional q-deformed oscillator algebra $H_q(1)$ is itself a quantum algebra associated with a Yang–Baxter equation. The generalization from a one-dimensional q-oscillator algebra to a multidimensional oscillator algebra $\mathcal{H}_q(n)$ is straightforward, since every component of the n-dimensional q-oscillator can be regarded as an independent one-dimensional q-oscillator.

It is also worth noting that there is an isomorphism between the SHO algebra and the differential operator algebra $\mathcal{D}(1)$, i.e. the algebra spanned by the operators x, ∂ and $x\partial$. So one may expect a q-deformed differential operator algebra to be isomorphic to the $\mathcal{H}_q(1)$ algebra. This is the $\mathcal{D}_q(1)$ algebra spanned by x, D and $x\partial$. The q-differential operator D is just the D operator proposed in [8] and defined via its action on C^{∞}, i.e.

$$Df(x)=\frac{f(qx)-f(q^{-1}x)}{(q-q^{-1})x}. \tag{21}$$

In [26], the D operator is expressed as an integral operator. We point out that, however, it can also be expressed as a formal function of the differential operator

$$D=\frac{1}{x}[x\partial]_q \tag{22}$$

and when $q\to 1$ this operator reduces to an ordinary differential operator.

The q-deformed differential operator algebra has the following commutation relations which are apparently isomorphic to those of $\mathscr{H}_q(1)$:

$$[D, x] = [x\partial + 1]_q - [x\partial]_q$$
$$[x\partial, x] = x \qquad [x\partial, D] = -D. \tag{23}$$

It is obvious that this algebra can be given a Hopf structure analogously to the above discussion and is therefore a quantum enveloping algebra of $\mathscr{D}(1)$. The generalization to multimode q-deformed differential operator algebra $\mathscr{D}_q(n)$ is easy.

Finally we would like to mention that the representation of $\mathscr{H}_q(1)$ or $\mathscr{D}_q(1)$, especially in the case of q being roots of unity, is an interesting topic still in progress.

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